$$- \alpha^2 A^2 b \int_{2\pi x_o}^{2\pi \tilde{x}_1} \frac{\cos^3 u \, du}{1 + A \cos u} \tag{B8}$$

But

$$\cos u = \frac{(1 + A \cos u) - 1}{A} \tag{B9}$$

then

$$\frac{\cos u}{1 + A \cos u} = \frac{1}{A} - \frac{1}{A(1 + A \cos u)} \tag{B10}$$

$$\frac{\cos^2 u}{1 + A \cos u} = \frac{1 + A \cos u}{A^2} - \frac{2}{A^2} + \frac{1}{A^2(1 + A \cos u)} \tag{B11}$$

$$\frac{\cos^3 u}{1 + A \cos u} = \frac{(1 + A \cos u)^2}{A^3} - \frac{3(1 + A \cos u)}{A^3} + \frac{3}{A^3} - \frac{1}{1 + A \cos u} \tag{B12}$$

Therefore

$$\int \frac{\cos u \, du}{1 + A \cos u} = \frac{u}{A} - \frac{1}{A} \int \frac{du}{1 + A \cos u} \tag{B13}$$

$$\int \frac{\cos^2 u}{1 + A \cos u} = \frac{\sin u}{A} - \frac{u}{A^2} + \frac{1}{A^2} \int \frac{du}{1 + A \cos u} \tag{B14}$$

$$\int \frac{\cos^3 u}{1 + A \cos u} = u \left(\frac{1}{A^3} - \frac{1}{2A} \right) - \frac{\sin u}{A^2} + \frac{\sin 2u}{4A} - \frac{1}{A^3} \int \frac{du}{1 + A \cos u} \tag{B15}$$

Define

$$I_o = \int \frac{du}{1 + A \cos u} \tag{B16}$$

Then

$$I_2 = \Big[I_o(\beta - 1)[\alpha^2(1 - A^2) - 1] + \left(\frac{1}{3}\beta + \alpha^2 + \frac{1}{6}\alpha^2 A^2 \beta - \alpha^2 \beta \right) u + \alpha^2 A(\beta - 1) \sin u + \frac{\alpha^2 A^2 \beta}{12} \sin 2u \Big]_{2\pi \tilde{x}_o}^{2\pi \tilde{x}_1} \tag{B17}$$

Since $A < 1$, the integral I_o can be obtained in closed form simply by replacing a by 1 and b by A in Equation ($A19$). This gives

$$I_o = \frac{2}{\sqrt{1 - A^2}} \tan^{-1} \left(\frac{(1 - A) \tan \pi \tilde{x}}{\sqrt{1 - A^2}} \right) \tag{B18}$$

Equations ($B3$), ($B17$), and ($B18$) then yield Equation (97).

Optimal Control of a Distillation Column

C. B. BROSILOW and K. R. HANDLEY

Optimal feedback control has been implemented on a fifteen tray pilot scale rectifying column. The results show that excellent control is obtained in spite of major upsets in the feed flow rate and for large changes in the controller set point. Design and implementation costs for the optimal control system should be competitive with those for standard control systems.

This paper is the first in a projected series of papers on the optimal control of countercurrent separation processes. Binary distillation has been chosen as the first type of countercurrent process to be studied because it presents the fewest difficulties in the design and implementation of the control system. For example, in a binary system, the dynamic state of the process can be estimated by measuring the temperature on every tray. This is an easy operation in comparison to measuring compositions which are required in an absorber or extractor. To make matters easier, this study deals only with the control of the overhead composition in a rectifying section of a column. Work is presently going on to extend results to a full column, and no special problems are anticipated.

Normally the control objective in distillation is to keep the process at the desired steady state in spite of disturbances, or to move from one desired state to another as quickly as possible. One possible mathematical formulation of this objective is the integral squared error criterion:

$$\min_{m(t)} \int_0^\infty [x' Qx + mRm]\, dt \tag{1}$$

subject to

$$b \leq m(t) \leq a \tag{2}$$

where the linearized process equations are

$$\dot{x} = Ax + Bm \tag{3}$$

the order of the process is the same as the number of stages.

The problem posed above has no simple yet rigorous feedback control solution. It does however admit to two different approximate solutions that depend on the relative magnitudes of the Q and R matrices. Each of these solutions gives a control law which depends in a simple way on the state of the process $x(t)$ and is thus easily implemented.

If the cost of control, which enters the problem formulation through the matrix R, is significant when compared with the value of the product, then the constraints on the control effort, $m(t)$, can be neglected. With the constraints neglected, an optimal control law of the following form results (*1*):

$$m(t) = \sum_{n=1}^{N} \alpha_n x_n(t) \equiv \alpha' x(t) \tag{4}$$

The constants α_n depend on the column parameters through the relations

$$\alpha = -PR^{-1}B \tag{4a}$$

and

$$PA + A'P - PBR^{-1}B'P = -Q \tag{4b}$$

Equation (4*b*) is commonly called a matrix Riccati equation for the $N \times N$ symmetric matrix P.

In practice the control law given by (4) is modified so that

$$\begin{aligned} m(t) &= \alpha' x(t) \quad \text{if} \quad b \leq \alpha' x \leq a \\ &= a \text{ or } b \text{ otherwise} \end{aligned} \tag{4c}$$

Whenever the control $m(t)$ is at the constraints (a or b) the control law given above is no longer optimal.

The control law (4*c*) has been applied to a number of chemical plant simulations by Lapidus (*2*, *3*). He observed that as the magnitude of the R matrix decreases, the controlled system becomes progressively more oscillatory. Thus, in cases where the cost of control is actually negligible, a fictitious R matrix must be introduced in order to achieve a well behaved control system (*3*).

An alternate approach, when the cost of control is negligible, is to retain the control effort constraints in the problem formulation and to seek a solution to Equation (1) with the matrix R set equal to zero. An approximate solution to this problem has been obtained by a number of investigators (*4*, *5*). Their solutions are essentially the same although their starting points for the problem and their insight into the significance of the solution are different.* The control law is written:

$$\begin{aligned} m(t) &= a \quad \text{if } \beta' x < 0 \\ &= b \quad \text{if } \beta' x > 0 \end{aligned}$$

$$\beta' x = \sum_{n=1}^{N} \beta_n x_n \tag{5}$$

The constant vector β depends on the process parameters according to

$$\beta = PB \tag{5a}$$

$$A'P + PA = -Q \tag{5b}$$

Again, P is a $N \times N$ symmetric, positive definite matrix. However, the matrices defined by Equations (4*b*) and (5*b*) are different (even though they are denoted by the same letter) and (4*b*) does not reduce to (5*b*) as the matrix R approaches zero.

The control law used in this study is that given by Equations (5), (5*a*), and (5*b*). It seems to be the more natural one for distillation column control since the cost of control is usually small when compared to the cost of off specification product.

An important property shared by both of the above control laws is that they are unaffected by errors in the estimation of the volumetric liquid holdup in the column. This is important because it is difficult to measure the liquid holdup under column operating conditions and because the holdup is often very sensitive to changes in vapor velocity in the column.

That the control law coefficients (α and β) are not affected by errors in estimating the liquid volume on a tray can be seen from the following: Assume the same volume holdup is on each tray. An error in estimating this volume has the effect of multiplying the right hand side of Equation (3) by some number, say $1/\epsilon$. This error in the A and B matrices propagates to the P matrix through Equation (5*b*) or (4*b*) so that $\tilde{P} = \epsilon P$ where $\tilde{P}$ is the incorrect P matrix. The vector of coefficients $\tilde{\beta}$ is given by the product

$$\tilde{\beta} = \tilde{P}\,\tilde{B} = \epsilon\,\frac{1}{\epsilon} PB = \beta$$

and so is independent of the original error. The same argument applies to the vector α in Equation (4*a*).

The control law given by Equation (5) has the additional property that it is not affected by a scale factor error in the estimation of the vector B. This is because the control law operates on the sign of the quantity $\beta' x$ rather than its magnitude.

EXPERIMENTAL SYSTEM

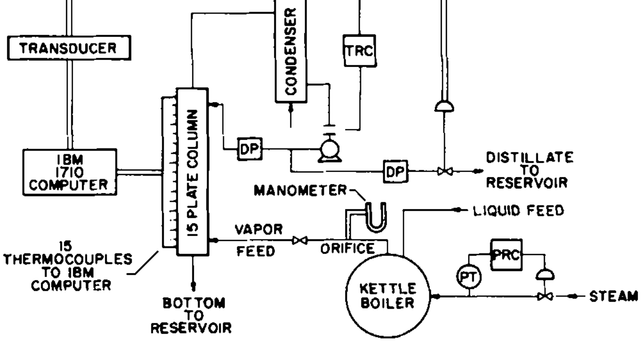

Fig. 1. Distillation column and auxiliaries.

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* A more concise derivation of the control law than that given in either (*4*) or (*5*) can be obtained by writing the first author.

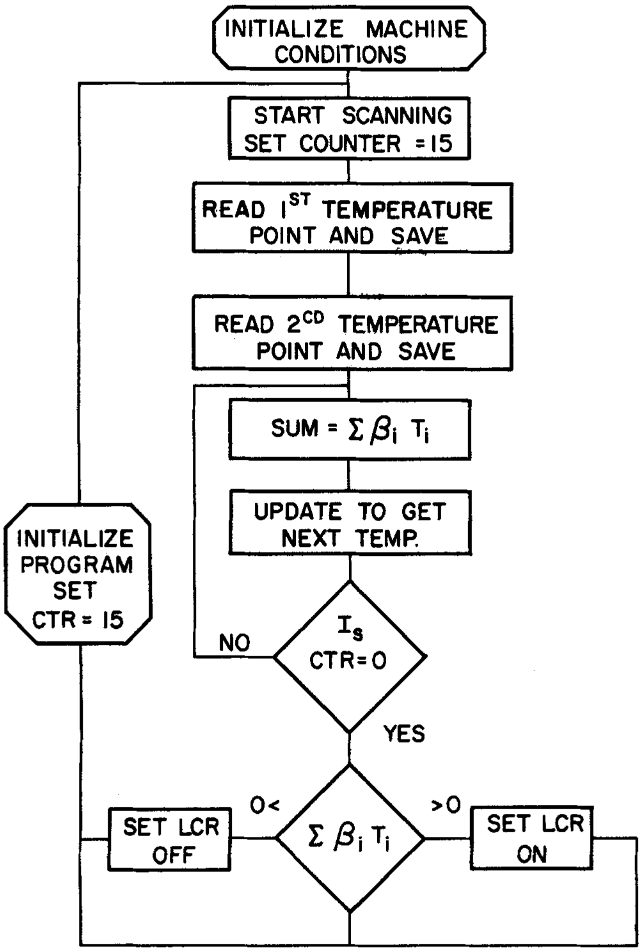

Fig. 2. Computer flowsheet for control law.

The distillation column used in this study is part of the computer control laboratory at Case Institute of Technology [6]. Figure 1 is a schematic of the distillation column, the computer and accessories. The column is 5 in. in diameter and has fifteen trays, three bubble caps per tray.

The binary system used for all experiments was ethylene glycol monomethylether (methylcellosolve) and water. This system was chosen for its low toxicity and high flash point.

Vapor was fed to the bottom of the column from a 50 gal. steam jacketed kettle. Approximate step changes in vapor feed rate were obtained by increasing the steam pressure in the kettle. The steam pressure to flow rate time constant was about 10 sec.

The vertical condenser was operated so as to produce a reflux temperature of 75°F. This is a considerable amount of subcooling, and the internal reflux rate was adjusted to account for it in all steady state calculations.

Iron-constantion thermocouples were used to measure the liquid temperature on each tray. They were run directly into the temperature compensated jacks on the IBM 1710 computer. The noise level in the thermocouple-computer circuit was estimated to be ± 0.3°F. [7].

The computer flow sheet required to implement the control law given by (5) is shown in Figure 2. The computer opens or closes a latch contact relay (LCR) depending upon the sign of the quantity $\beta'x$. This relay in turn operated a current to air transducer putting out a 3 to 15 lb. air signal to operate the distillate control value (Figure 1).

PROCESS MODEL

The column model was obtained by linearizing the equation resulting from the following assumptions:

1. Negligible vapor holdup in the column.
2. Perfect mixing of liquid on a tray.
3. Adiabatic column operation.

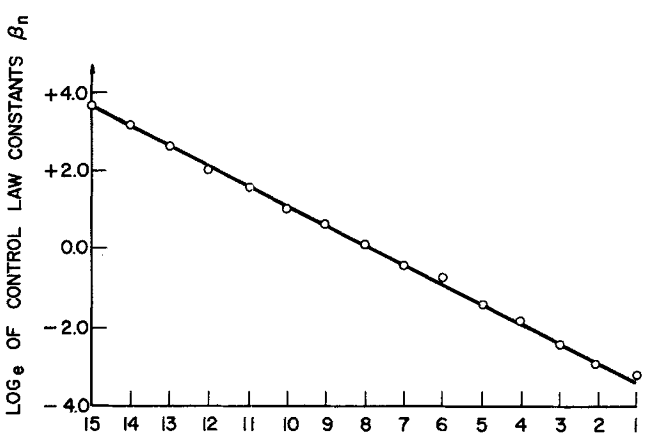

Fig. 3. Control law constants β_n vs. plate number.

4. Instantaneous attainment of the pseudo equilibrium between liquid and vapor phases.
5. No change in the liquid holdup on a tray due to perturbations in liquid or vapor flow rates.
6. Equal molal overflow.

Under these assumptions (numbering the top tray as 15 and the bottom tray as 1) the column model is:

$$\left(\overline{M}_n + \frac{\partial \overline{M}_n}{\partial x_n}\overline{z}_n\right)\frac{dx_n}{dt} = (\overline{L})\,x_{n+1} + (\overline{V}K_{n-1})x_{n-1}$$

$$- (\overline{L} + \overline{V}K_n)x_n + (K_{n-1}\overline{z}_{n-1} - K_n\overline{z}_n)v + (\overline{z}_{n+1} - \overline{z}_n)\mathcal{L}$$

$$n = 1 \ldots 15. \quad (6)$$

The last assumption is valid for the cellosolve-water system only at steady state. Because of the variation of liquid holdup with composition due to the different molar densities of water (3.29 lb. moles/cu.ft. at 101°C.) and cellosolve (0.832 lb. moles/cu.ft. at 101°C.) this assumption does not hold during a transient as shown in Appendix 1.

Subsequent calculations have shown that the inclusion of this assumption introduces negligible error into the control law coefficients, β_n, even though the time constant of the process is in error by about 100%.

The control law coefficients, β_n, are plotted vs. plate number in Figure 3. The exponential dependence of the coefficients on plate number seems to be characteristic of those countercurrent separation processes in which the control objective is to maintain a given overhead product composition by manipulation of the reflux. The numerical method used to compute the control coefficients is detailed in Appendix 2. Values for the various parameters needed in the calculation were taken from the assumed pseudo

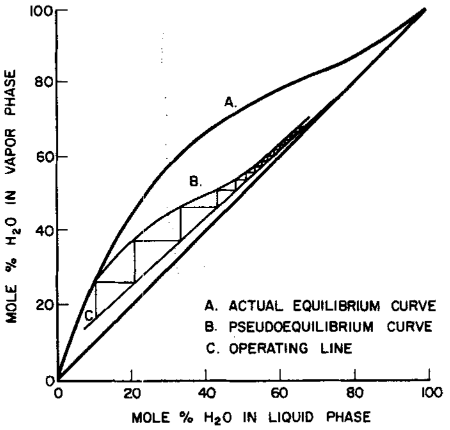

Fig. 4. Assumed column operating conditions for controller design.

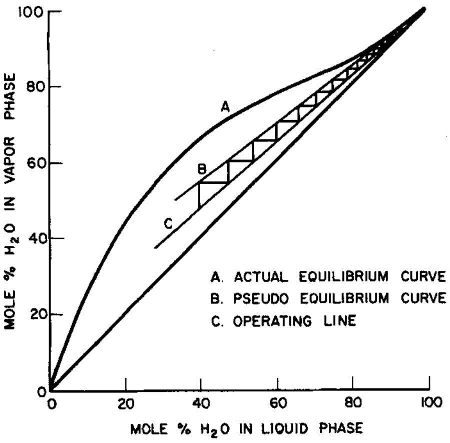

Fig. 5. Actual column operating conditions.

equilibrium curve, and column operating conditions shown in Figure 4. The actual column operating conditions as obtained from refractometric measurements of the liquid composition on each plate are shown in Figure 5. This unintentional error introduces an error of about 50% in the slope of the β vs. plate number curve in Figure 3. That is, the slope should have been $\frac{1}{4}$ rather than $\frac{1}{2}$ as shown. However, subsequent experiments on a numerical simulation of the process indicates that the response of the system does not change significantly for values of the slope ranging from 1 to $\frac{1}{8}$. This is as it must be; a practical control system has to be capable of operating satisfactorily away from the region for which it was designed, and for values of the control parameters which are not strictly optimal.

RESULTS

Figures 6 through 9 show the response of the controlled [Equation (5)] and uncontrolled process to set point and feed flow disturbances. The temperature response, rather than the composition response is shown because temperature was the variable actually under control. The temperature controller used the coefficients shown in Figure 3. [Over the operating range of the column there was no significant difference between the coefficients for composition or temperature control (7).] Composition control would have required that the temperature measurements be combined with measurements of the column pressure which varied randomly as the ambient pressure changed. Such refinements were considered unnecessary in these initial investigations. Also, the temperature on the second tray from the top is shown rather than the top tray temperature, as the mixing of cold reflux with the liquid on the top tray caused above average temperature fluctuations which were not indicative of composition changes.

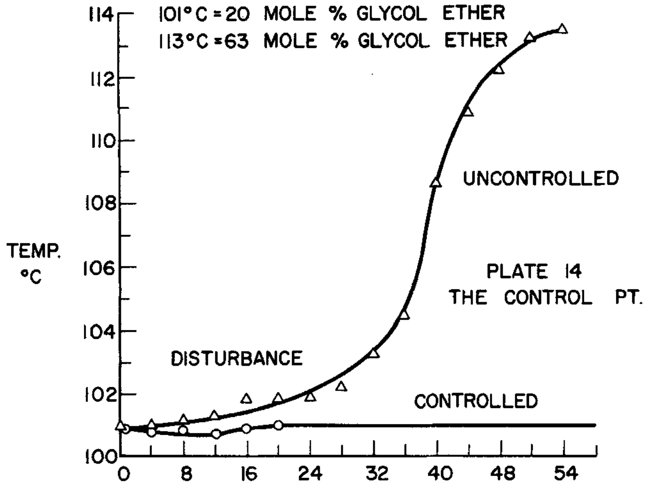

Fig. 6. Controlled and uncontrolled response to a step change in vapor feed rate from 10 to 15 cu. ft./min.

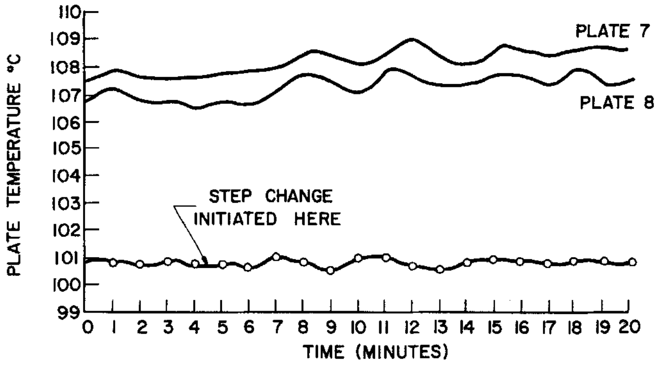

Fig. 7. Response of column to step change in vapor feed rate of 10 to 15 cu. ft./min.

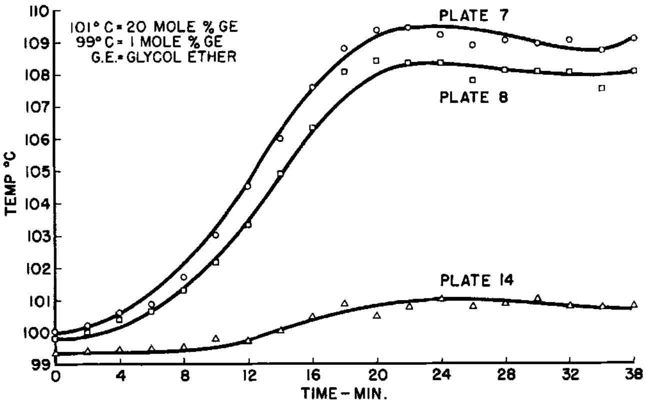

Fig. 8. Response of column to change from total reflux set point. Vapor feed = 10 cu. ft./min.

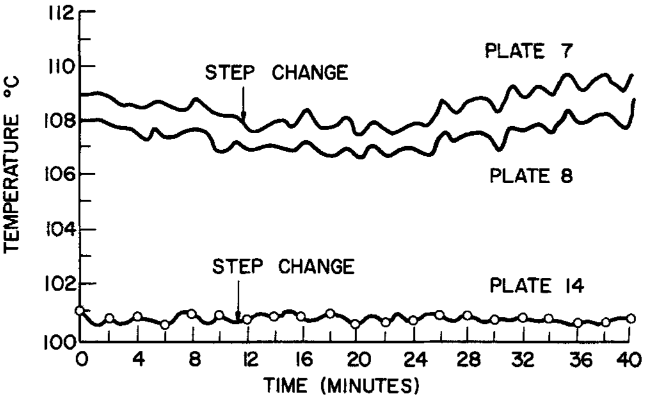

Fig. 9. Column response to change in vapor feed rate of 10 to 33 cu. ft./min.

The small oscillations in temperature evidenced in Figures 6 through 9 are probably due to the lack of resolution in the thermocouple circuitry. The noise level in this circuitry was at least ± .3°F. which is about the magnitude of the cycles.

As expected, there was no steady state error in the overhead composition due to changes in vapor feed rate. What happens is that when steady state is reached, the control law acts to maintain this condition by switching between limits so that

$$\underset{t}{\text{average}}\ \beta' x(t) = 0 \qquad (7)$$

One way for Equation (7) to be satisfied is for the average, over time, of each $x_n(t)$ to be zero. This condition actually occurs because the control can reestablish the L/V ratio which existed before the disturbance and thereby return all tray compositions to their original levels.

Ideally, one would like the rate of switching at steady state to be high enough for the valve to act as a filter for the switches and thereby assume the correct steady state position. Such behavior was observed in analog simulations [5] but in this series of experiments the valve showed sustained oscillation from opened to closed with a period of about ½ min. This implies that in practice some form of digital filtering of the control signal will be necessary.

To complete the discussion we will analyze what happens when the desired overhead composition remains the same, but the feed concentration changes. The control law in the computer is assumed to remain unchanged so that at steady state

$$\text{average} \sum_{n=1}^{N} \beta_n z_n = \sum_{n=1}^{N} \beta_n \bar{z}_n \qquad (8)$$

Equation (8) is simply Equation (7) written in terms of actual rather than perturbation variables. Since β and $\bar{z}_n$ are unchanged, the right hand side of Equation (8) is constant. However, $z_1 \neq z_1$ because of the change in feed concentration. Therefore, the entire profile must readjust. For example, consider the 7 plate column operating as shown in Figure 10. The control law constants are taken to be the same as those for the first 7 plates of our 15 plate column (Figure 3). (A 7 plate column was chosen over our 15 plate column because the McCabe-Thiele diagram is easier to work with.) The desired overhead composition is 0.90. Now, the feed concentration is changed from $y_f = 0.505$ to $y_f = 0.40$. After an initial transient the column settles down to the profile shown in Figure 11. The overhead composition is now about 0.905 or about 0.5% higher than before the change in feed concentration. If the feed concentration had increased rather than decreased then the overhead composition would be lower than 0.90. The explanation of this behavior is that in attempting to keep the weighted sums

$$\text{of the column composition profile} \left[\text{that is, } \sum_{n=1}^{N} \beta_n z_n \right]$$

the same, the control law forces the top compositions to move in a direction opposite to the feed change. Thus, in Figure 11 the reflux ratio is considerably higher than that in Figure 10. Note, however, that because of the relatively high weights assigned to the composition on trays near the top of the column, the amount of off set is very slight.

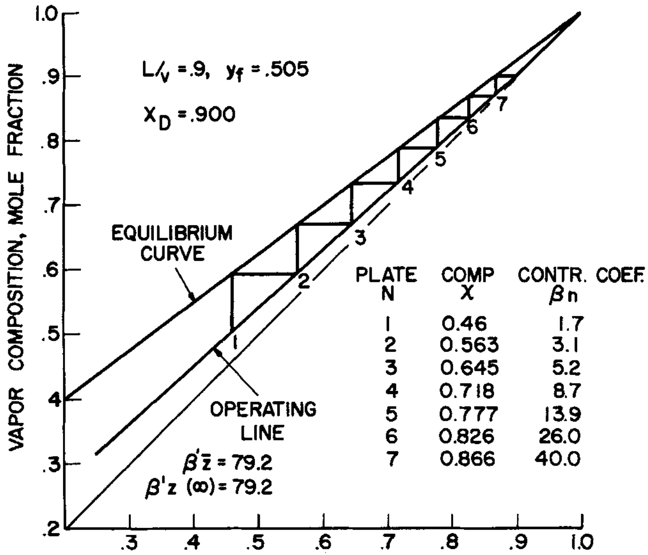

PLATE N	COMP X	CONTR. COEF. βn
1	0.46	1.7
2	0.563	3.1
3	0.645	5.2
4	0.718	8.7
5	0.777	13.9
6	0.826	26.0
7	0.866	40.0

Fig. 10. Composition profile before disturbance in feed concentration.

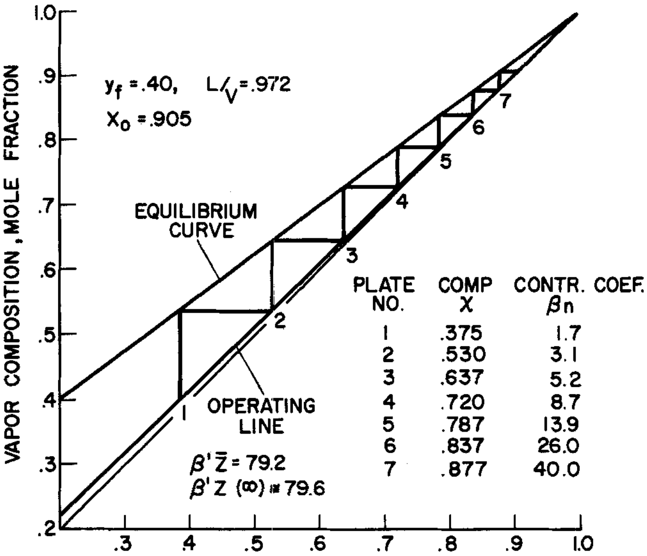

PLATE NO.	COMP X	CONTR. COEF. βn
1	.375	1.7
2	.530	3.1
3	.637	5.2
4	.720	8.7
5	.787	13.9
6	.837	26.0
7	.877	40.0

Fig. 11. Composition profile after disturbance in feed concentraton.

CONCLUSIONS

The control system seems to be ideally suited to its task; it performs well in spite of model inaccuracies, it behaves well over wide ranges of column operation, and it exhibits relatively little off set under load changes.

Future publications will deal with the control of a full column and with techniques for reducing the number of measurements required by the control system.

ACKNOWLEDGMENT

The authors wish to express their appreciation to Professor D. M. Young for the use of his very fine eigenvector program.

NOTATION

a, a_i, b, b_i = scalar constraints on the control effort
A = $N \times N$ process matrix
B = matrix of process constants
CTR = counter
h_n = enthalpy of the liquid on tray n
H_n = enthalpy of the vapor from tray n
$K = (k_{ij}) = N \times N$ matrix with elements k_{ij}
K_n = the slope of the pseudo equilibrium curve at $z_n = \bar{z}_n$
$\mathcal{L}$ = deviation of liquid flow rate about steady state
L = liquid flow rate
LCR = latch contact relay
$m(t)$ = deviation of reflux flow rate about the steady state
M_n = liquid holdup on tray n
N = number of trays
P = $N \times N$ Riccati matrix
q_n = heat leak on stage n
$Q = (q_{ij}) = N \times N$ positive semidefinite weighting matrix
R = 1×1 weighting matrix
t = time
T = $N \times N$ matrix of eigenvectors
U_n = internal energy of the liquid on tray n
v = deviation of vapor flow rate about steady state
$V(t)$ = Lyapunov function

$x(t) = (x_1(t) \ldots x_N(t))$ = a vector with components $x_n(t)$

$x_n(t)$ = deviation of liquid composition on tray n from steady state

$z_n(t)$ = liquid composition on stage n

$y_n(t)$ = vapor composition on stage n

Greek Letters

α, β = vectors, $\alpha = (\alpha_1 \ldots \alpha_N)$, $\beta = (\beta_1 \ldots \beta_N)$

α_n, β_n = control law coefficients

∇ = forward difference operator $\nabla y_n = y_{n+1} - y_n$

ϵ = a positive number

$\lambda = (\lambda_{ii})$ = diagonal matrix of eigenvalues λ_{ii}

Superscripts

$'$ = vector transposed

$\sim$ = denotes a variable which contains an error

$-$ = denotes a constant which is equal to the value of the variable at steady state

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APPENDIX 1

Unsteady State Material and Energy Balances

The column mass and energy balances are

Mass
$$\frac{dM_n z_n}{dt} = \nabla\{L_n z_n + V_{n-1} y_{n-1}\}; \quad n - 1 \ldots N \tag{A3.1}$$

$$\frac{dM_n}{dt} = \nabla\{L_n + V_{n-1}\}; \quad n = 1 \ldots N \tag{A3.2}$$

Energy
$$\frac{dU_n M_n}{dt} = \nabla\{L_n h_n + V_{n-1} H_{n-1}\} - q_n; \quad n = 1 \ldots N \tag{A3.3}$$

Equilibrium
$$y_n = f(z_n) \tag{A3.4}$$

According to assumption 3 $q_n = 0$, $n = 1 \ldots N$. The classical steady state assumption of constant molar overflow is equivalent to the assumption that H_n and h_n are constant for all n. Using these facts and the added assumption that $h_n = U_n$ we get from Equations (A3.2) and (A3.3) that

$$h \nabla (L_n + V_{n-1}) = \nabla (h L_n + H V_{n-1})$$

which implies that $\nabla V_n = 0$ or V_n = a constant. This fact eliminates the need for Equation (A3.3). Equation (A3.1) and (A3.2) reduce to

$$\frac{dM_n z_n}{dt} = \nabla L_n z_n + V \nabla y_{n-1}; \quad n = 1 \ldots N \tag{A3.1a}$$

$$\frac{dM_n}{dt} = \nabla L_n; \quad n = 1 \ldots N \tag{A3.2a}$$

Since $M_n = M_n(z_n)$, $\dfrac{dM_n}{dt}$ is not zero and L_n is not a constant over n except at steady state.

APPENDIX 2

Computation of the Control Law

In order to obtain the coefficients β_n of the control law given by Equation (5), we must solve the Riccati Equation (5*b*) for the matrix P.

$$A'P + PA = -Q \tag{5b}$$

The technique suggested here and used in [7] is to transform (5*b*) into an equivalent equation which is easy to solve analytically. No claim is made concerning the computational efficiency of the algorithm. Actually, computational efficiency is not very important since the entire computation is performed only once or twice.

We start by assuming that we can obtain the matrix of eigenvectors of the matrix A in (5*b*). Let this matrix be denoted by T:

$$AT = T\lambda \tag{A2.1}$$

where λ = diagonal matrix of eigenvalues.

We further assume that

1. The process described by Equation (3) is stable so that all eigenvalues of A have negative real parts.
2. Every eigenvalue of A is distinct.

The above properties are always satisfied by binary distillation columns under our assumptions.

Now define the matrix K

$$K \equiv T'PT \tag{A2.2}$$

so that

$$P = (T')^{-1} K T^{-1} \tag{A2.3}$$

substituting (3) into (5*b*) gives

$$A'(T')^{-1} K T^{-1} + (T')^{-1} K T^{-1} A = -Q$$

which reduces to

$$(\lambda T^{-1})' K T^{-1} + (T')^{-1} K (\lambda T^{-1}) = -Q$$

and finally to

$$\lambda K + K\lambda = -T'QT \tag{A2.4}$$

which in scalar form is

$$\lambda_{ii} k_{ij} + k_{ji} \lambda_{jj} = -(T'QT)_{ij}$$

Since K is symmetric, as can be seen by transposing both sides of A2.4, then

$$k_{ij} = -(T'QT)_{ij} / (\lambda_{ii} + \lambda_{jj}) \quad i,j = 1 \ldots n. \tag{A2.4}$$

The matrix P can now be constructed from A2.3. It is relatively easy to show that the matrix P is positive definite [7]. In this work it also turned out to be strictly positive (that is, $P_{ij} > 0$; $i,j = 1 \ldots n$). This is probably no coincidence since the matrix A has some very special properties (for example, $A^{-1} > 0$; $\lambda(A) < 0$). The matrix Q used in calculating the control law was as follows:

$$q_{ij} = 0$$

for all i, j except

$$q_{14,14} = 1$$

The subscripts *14* refer to the fact that the composition on the tray which is second from the top is the one which is to be maintained constant. The second from the top tray was chosen over the more normal choice of the top tray because the subcooled reflux made temperature measurement on the top tray a relatively unreliable indicator of composition.